Math 217 Fall 2025

Quiz 15 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces and $T\colon V \to W$ is a linear transformation. The *kernel* of T is ...

   > **Solution:** The set of vectors in V that T sends to the zero vector of W:
   >
   > $$\ker T \;=\; \{\, v \in V \;:\; T(v) = 0_W \,\}.$$

   (b) An *isomorphism* of vector spaces is ...

   > **Solution:** A bijective linear transformation. Equivalently, a linear map $T\colon V \to W$ that has a (necessarily linear) inverse $T^{-1}\colon W \to V$. If such a map exists, we say V and W are *isomorphic* and write $V \cong W$.

   (c) To say that a list of vectors $(x_1, x_2, \ldots, x_d)$ in a vector space X is *linearly dependent* means ...

   > **Solution:** There exist scalars $a_1, \ldots, a_d$, not all zero, such that
   >
   > $$a_1x_1 + \cdots + a_dx_d \;=\; 0_X.$$

2. Let V and W be vector spaces, and suppose $T : V \to W$ is a linear transformation of vector spaces *with the same (finite) dimension*. Show that T is surjective if and only if T is injective.

> **Solution:** Suppose $n = \dim V = \dim W < \infty$. By the Rank–Nullity Theorem,
>
> $$\dim V \;=\; \dim\ker(T) + \dim\ker(T) \;=\; n.$$
>
> If T is injective, then $\dim\ker(T) = 0$, so $\dim\operatorname{im}(T) = n$. Hence the image of T is an n-dimensional subspace of W; since $\dim W = n$, the image must equal W, i.e., T is surjective.
>
> Conversely, if T is surjective, then $\dim\operatorname{im}(T) = \dim W = n$, so $\dim\ker(T) = n - \dim\operatorname{im}(T) = 0$. Thus the kernel is $\{0\}$ and T is injective.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) The zero vector is a basis for the vector space $\{\vec{0}\}$.

Solution: FALSE. A basis must be linearly independent. The singleton $\{0\}$ is linearly *dependent* because $1 \cdot 0 = 0$ with a nonzero coefficient. The correct basis for 0 is the *empty set*, which is linearly independent and spans 0.

(b) The kernel of the trace map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ has dimension $n^2 - n$.

Solution: FALSE. The trace map $\mathrm{tr}\colon M_n(\mathbb{R}) \to \mathbb{R}$ is linear and nonzero, so $\dim \mathrm{im}(\mathrm{tr}) = 1$. By Rank–Nullity,

$$\dim \ker(\mathrm{tr}) \;=\; \dim M_n(\mathbb{R}) - \dim \mathrm{im}(\mathrm{tr}) \;=\; n^2 - 1,$$

not $n^2 - n$. Indeed, $\ker(\mathrm{tr}) = \{A \in M_n(\mathbb{R}) : \mathrm{tr}(A) = 0\}$ is an $(n^2 - 1)$-dimensional subspace of $\mathbb{R}^{n\times n}$.